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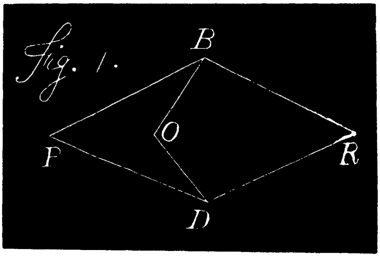
THE PEAUCELLIER MACHINE AND OTHER LINKAGES.

BY PROF. WM. WOOLSEY JOHNSON, ST. JOHN'S COLLEGE, ANNAPOLIS, MD.

In proposing the following problem in the Mathematical department of the Educational Times — "*If two opposite angles of a jointed rhombus move in the same fixed circle, and one of the other angles move in a second fixed circle passing through the centre of the first, prove that the fourth angle will move in a straight line*" — Professor J. J. Sylvester remarks "this is the admirably simple solution (discovered within the last few years by the French Geometer Peaucellier) of a problem that had generally been supposed as ineffectuable as the quadrature of the circle; and the assumed impossibility of which many mathematicians of the first order had been long laboring to demonstrate; viz., the rigorously exact commutation of circular with rectilinear motion by means of a combination consisting exclusively of rods, joints and points."

The following particulars with regard to this first solution of the problem of "Parallel Motion," as it is called in Mechanics, are taken from the Transactions of the London Mathematical society, to which Prof. Sylvester communicated it in Nov., 1873. The fact of Peaucellier's discovery, (which was made in 1862), but no description of it, was communicated in 1867, by a brother officer of engineers, to the Paris Philomathic Society; but attracted no attention until 1871, when the mechanism was rediscovered by a young man named Lipkin, a pupil of the celebrated Tchebicheff of the University of St. Petersberg, who had himself been laboring to demonstrate the impossibility of that which his pupil thus achieved.

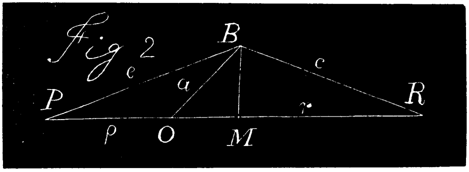


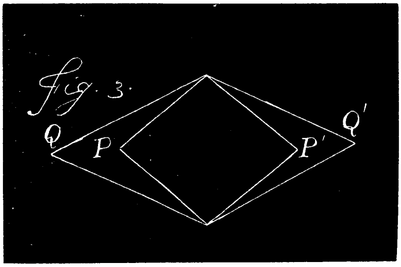
Professor Sylvester calls the aparatus consisting of the jointed rhombus, *PBRD*, (Fig. 1), and the two equal rods *BO* and *OD*, jointed at *O* with one another, and at *B* and *D* with two angles of the rhombus, a *Peaucellier cell*; and has succeeded in proving that any motion which can be produced by an aparatus of jointed rods, or linkage, can likewise be produced by a combination of Peaucellier cells. The curves thus described necessarily belong to the class of Algebraic curves, and Professor Sylvester has shown how to generate all such curves up to the 6th degree.

The fundamental property of the Peaucellier cell is that the points *P*, *O* and R are always in a straight line, and that the distances of P and R which are called *the poles* from O which is called the *fulcrum* are *inversely proportional.* This is readily seen as follows; denoting BP or BR by c, OB by a, OP by ρ and OR by r, and dropping the perpendicular BM which will of course bisect PR, we have

$$r\rho = (MR + OM)(MR - OM) = MR^2 - OM^2 = c^2 - a^2,$$

that is, the product of the variable distances of the poles from the fulcrum is constant. This constant $c^2 - a^2$ may be denoted by k^2 and is called the *modulus* of the cell. Fig. 1, in which r and ρ have opposite directions from O, represents a *negative* cell. If $a > c$, r and ρ will be in the same direction and the cell will be a *positive* one. For describing curves with discontinuous branches it is convenient to add two more bars, forming a rhombus within a rhombus, as in Fig. 3, where Q and Q' are the poles of a negative cell of which P or P' is the fulcrum and P, P' are the poles of a positive cell of which Q and Q' are the fulcra. This linkage of eight bars is called a complete Peaucellier cell.





From the relation $r\rho = \pm k^2$ it will be seen that, if the fulcrum O (see Fig. 1) be fixed, and the point P be made to describe any curve, the point R will describe the inverse of this curve, O being the centre of inversion. The polar equation of the curve in which P moves, referred to O as pole, is in fact a relation between ρ and θ in which we have only to substitute for ρ its value $k^2 \div r$ (or $-k^2 \div r$ for the negative cell), to produce the relation between r and θ which is the polar equation of the inverse curve described by R.

To produce rectilinear motion, let P be constrained to move in a circle passing through O, by a bar PC (not represented in the figure) fixed at C, so that $OC = PC = d$. Then, OC being the initial line, the equation of the path of P is

$$\rho = 2d \cos\theta,$$

and R will describe $\dfrac{-k^2}{r} = 2d\cos\theta,$ or $r = \dfrac{-k^2}{2d}\sec\theta,$

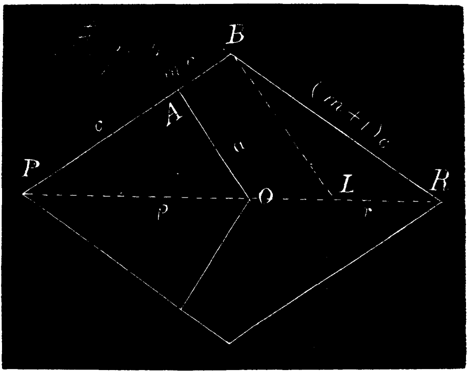
a straight line perpendicular to OC. The length of path described by R will depend upon the range of the variable r, of which the limiting values are the sum and the difference of the rods OB and BR.

Thus in Peaucellier's solution of the problem of "parallel motion" a linkage of seven bars attached to two fixed centres is employed.

If P move in a circle not passing through O, R will describe a circle; for the two radii vectores, ρ and ρ', corresponding to the same value of θ in a circle have a constant product, hence r is directly proportional to ρ', and will describe a similar curve, that is a circle.

The geometrical proposition proved above in connection with Fig. 2 may be stated thus:—*In the two triangles which can be formed with given values of two of the sides and the angle opposite one of them, the product of the third sides is equal to the difference of the squares of the given sides.*

Another linkage proposed by Professor Sylvester in a problem (Educational Times, Feb. 1874) consists of a rhombus in which points on adjacent sides equally distant from one angle are connected by means of equal rods with the fulcrum, as in Fig. 4. Denoting AO by a, PA by c and AB by mc, a side of the rhombus equals $(m+1)c$. Denote as before OP by ρ and OR by r, and draw BL parallel to AO. Then $OL = m\rho$, $PL = (m+1)\rho$ and by the above proposition $LR = (m+1)(c^2 - a^2) \div \rho$; therefore



$$r = m\rho + (m+1)\frac{c^2 - a^2}{\rho},$$

or putting the constant $(m+1)(c^2 - a^2) = k^2$,

$$r = m\rho + \frac{k^2}{\rho}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

If $c = a$, $k^2 = 0$, $r = m\rho$, and P and R describe similar curves, of which the linear scales are as $1 : m$.

When c is not equal to a, the variable ρ has a range between $c + a$ and $c \sim a$, and cannot pass through zero; but r can pass through zero and change sign if

$$\frac{m+1}{m}(a^2 - c^2) > 0,$$

for this is the value of ρ^2 which makes $r = 0$. On the other hand r cannot change sign, but has a minimum if

$$\frac{m+1}{m}(c^2 - a^2) > 0,$$

for this last is the value of ρ^2 which makes $\frac{dr}{d\rho}=0$, [in the figure where m is positive and $c>a$, the latter is the possible case] and the value of the minimum is $2\sqrt{m}.k$.

Equation (1) may be put in the form $m\rho^2-\rho r+k^2=0$, if now R be constrained to move in the circle

$$r=2d\cos\theta \,.\,.\,.\,.\,.\,.\,.\,.\,.\,.\,.\,.\,.\,.\,.\,. \quad (2)$$

passing through the point O, then P will describe

$$m\rho^2-2d\rho\cos\theta+k^2=0, \,.\,.\,.\,.\,.\,.\,.\,.\,.\,.\,.\,. \quad (3)$$

a circle whose centre is at the distance $d\div m$ from O, and whose radius is

$$\sqrt{\left(\frac{d^2}{m^2}-\frac{k^2}{m}\right)}=\sqrt{\left[\frac{d^2}{m^2}+\frac{m+1}{m}\left(a^2-c^2\right)\right]}.$$

In the first of the cases mentioned above, this radius is greater than $d\div m$, so that O is within the circle described by P, and R will describe the complete circle (2) while P describes either of the arcs into which the circle (3) is divided by the tangent to (2) at O.

In the second case, the circle (3) is impossible unless $2d>2\sqrt{m}.k$, and in that case P describes the complete circle (2) while R reciprocates over that arc of the circle (2) which lies farther from O than the minimum value of r, namely $2\sqrt{m}.k$.

Now let P be constrained to move in the circle

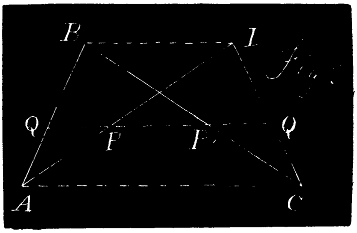
$$\rho=2d\cos\theta \,.\,.\,.\,.\,.\,.\,.\,.\,.\,.\,.\,.\,. \quad (4)$$

passing through O, P will describe the curve

$$r=2md\cos\theta+\frac{k^2}{2d}\sec\theta, \,.\,.\,.\,.\,. \quad (5)$$

a curve of the third degree. If the diameter $2d$ be that value of ρ which makes $r=0$, P will describe a cissoid having its cusp at O, this will of course be possible only in the first of the above cases: for instance, if $m=\frac{1}{3}$, $a=5$ and $c=4$; $k^2=-12$, and $\rho=6$ makes $r=0$: then making $2d=6$ we have the cissoid $\qquad r=2\cos\theta-2\sec\theta$.

In the Mathematical Messenger for October, 1874, Mr. Harry Hart has shown that four points having the relation of the points P, Q, Q', P' of Peaucellier's complete cell, Fig. 3, may be connected by a linkwork of four bars only. Mr. Hart employs two equal bars AB and DC jointed at their extremities to two other equal bars BC and AD, as in Fig. 5, so that AC and BD are parallel lines. Then taking points Q P P' Q' cutting the several bars in



the same ratio, it is evident that these points lie in a straight line parallel to AC or BD. Now since the triangles ABD and ADC have two sides of the one equal two sides of the other, the angles opposite one pair of sides equal and those opposite the other pair supplementary, the product of the third sides BD and AC is equal $BC^2 - AB^2$ and is therefore constant. Now QP bears a fixed ratio to BD and PQ' bears a fixed ratio to AC, therefore the product $QP \times PQ'$ is constant. Thus Q and Q' correspond to the poles of a negative cell of which P or P' is the fulcrum; and P and P' to the poles of a postive cell of which Q or Q' is the fulcrum. The modulus of the cell may be altered by adjusting the position of the points upon the bars, without the necessity of altering the lengths of the latter. Fixing a fulcrum, say Q, and adding a bar which shall cause P to move in a circle passing through Q, P' of course describes a straight line. Mr. Hart has thus succeeded in converting circular into rectilinear motion by a linkwork of five bars, and it can hardly be doubted that he has obtained the simplest possible solution.

Mr. Hart has also shown that if the points A and D be fixed, thus forming a linkwork of three bars, the point P' will describe a curve the inverse of which with respect to P is a conic. As four bars only are necessary to describe the inverse of a given curve, he describes the conic by a linkage of seven bars.

The locus of P' when A and D are fixed points may be found thus—Denote AD by $2a$, AB by $2b$, the distance of P from the middle point of AD by c, the distance from this point to the intersection of AD and BC by the variable z. Taking AD as initial line and P as pole, $PP' = \rho$ and angle $P'PD = \theta$.

Then $$AC = 2(a + z) \cos \theta,$$

and $$BD = 2(a - z) \cos \theta;$$

and since, as before remarked, $AC.BD = AD^2 - AB^2$,

$$(a^2 - z^2)\cos^2 \theta = a^2 - b^2,$$

or $$b^2 - a^2 \sin^2 \theta = z^2 \cos^2 \theta;$$

but $$z \cos \theta = \tfrac{1}{2}\rho - c \cos \theta,$$

eliminating z $$b^2 - a^2 \sin^2 \theta = (\tfrac{1}{2}\rho - c \cos \theta)^2.$$

The inverse with respect to P, modulus being $2k^2$, so that $\rho = 2k^2 \div r$, is

$$r^2(b^2 - a^2 \sin^2 \theta) = (k^2 - cr \cos \theta)^2$$

or, in rectangular co-ordinates,

$$(b^2 - c^2)x^2 + (b^2 - a^2)y^2 + 2ck^2x = k^4.$$

Since $a > b$ we have hyperbolas when $c < b$, parabolas when $c = b$, and ellipses when $c > b$.